Squaring this last equation

$$m^8+2m^7+(1-2A)m^6+(64+2B-2A)m^5+[160-64(a+b+c)+A^2$$
$$+2C+2B]m^4+[64(ab+ac+bc)-160(a+b+c)+100+2C-2B]m^3$$
$$+[160(ab+ac+bc)-100(a+b+c)-64abc+B^2-2AC]m^2$$
$$+[100(ab+ac+bc)+160abc+2BC]m+C^2-100abc=0.$$

This equation gives $m$. $m$ in (4) gives $s$, and $s$ in (1), (2), (3) gives $x$, $y$, $z$.

Also solved by *MARCUS BAKER*, who combines 169 and 173 under one solution.

170. Proposed by S. F. NORRIS, Baltimore City College, Baltimore, Md.

Find by strictly quadratic methods at least one set of values of $x$ and $y$ in the equations $x^2y^2+x=a=38$, and $xy+y^2=b=15$.

Comments and Analysis by MARCUS BAKER, Washington, D. C.

The values of $xy$ and of $x$ from the second equation substituted in the first one give an equation in $y$; and similarly the values of $xy$ and of $y^2$ from the second equation substituted in the first one give an equation in $x$. The two equations resulting from this elimination are

$$x^5+(b^2-a)x^4+2bx^3+(1-2ab)x^2-2ax+a^2=0....(1),$$
$$y^5\qquad -2by^3-\qquad y^2+(b^2-a)+b=0....(2),$$

equations of the fifth degree and of course insoluble by quadratics. Restoring numbers, the equations become

$$x^5+187x^4+30x^3-1139x^2-76x+1444=0....(3).$$
$$y^5\qquad -30y^3-\qquad y^2+187y+\ 15=0....(4).$$

It is obvious by inspection of the original equations that $x=2$ and $y=3$ are a pair of roots; therefore (3) is exactly divisible by $x-2$ and (4) is similarly divisible by $y-3$. Performing this division there results

$$x^4+189x^3+408x^2-323x-722=0....(5),$$
$$y^3+\ 3y^3-\ 21y^2-\ 64y-\ 5=0....(6),$$

equations of the fourth degree. Can these equations be solved by *strictly quadratic methods?* The answer depends on the meaning of the italicized words. Every quartic is solvable after the manner of quadratics in terms of an auxiliary involved in a cubic equation; and every cubic is solvable after the manner of quadratics in terms of an auxiliary involved in a quadratic equation; and every quadratic is solvable in terms of an auxiliary involved in a simple equation. In a certain sense, therefore, *every* biquadratic is solvable by quadratics. If by *strictly quadratic methods* it is implied that the auxiliary involved in the cubic is

excluded it only remains to apply the test for determining whether in the proposed equations (5) and (6), the auxiliary cubic is necessary. The general quartic

$$x^4+ax^3+bx^2+cx+d=0$$

is solvable by quadratics, without the intervention of a cubic, as shown by Thomas Simpson, in his Algebra, in three cases, viz., when

$$c=\tfrac{1}{2}af, \text{ or } a\sqrt{d} \text{ or } 2\sqrt{df},$$

where, for brevity, $f=b-\frac{1}{4}a^2$. Considering equation (6),

$$\begin{array}{ll} a=+3 & d=-4 \\ b=-21 & f=-23\tfrac{1}{4} \\ c=-64 & \end{array}$$

whence $\frac{1}{2}af=-34\frac{7}{8}$; $a\sqrt{d}=3\sqrt{-5}$; $2\sqrt{df}=5\sqrt{31}$.

As none of these quantities equal $c$, the conclusion is that $y$, and similarly $x$, cannot be determined by *strictly quadratic methods.*

Also solved by *J. E. SANDERS*, and *G. B. M. ZERR.*

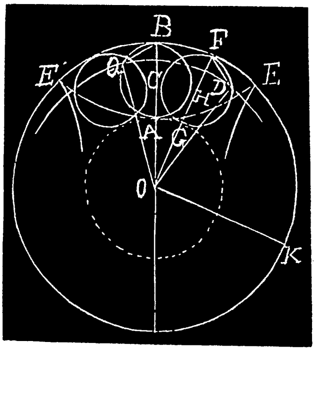
## GEOMETRY.

**191.** Proposed by J. V. ADAMS, St. Louis, Mo.

Trisect any angle by means of the hypocycloid.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelyhia, Pa.; JOHN J. QUINN, Warren High School, Warren, Pa.; and M. E. GRABER, Heidelberg University, Tiffin, O.

Let $EAE'$ be an arc of the hypocycloid from cusp to cusp; $R$=radius of fixed circle; $r$=radius of generating circle; $AQB$ the central generating circle; $C$ its center; $Q$ any point on this circle. Join $QB$, $QO$. With $QO$ as a radius and $O$ as a center describe the arc $QD$ meeting the hypocycloid in $D$. Let $GDF$ be the generating circle when $D$ is the generating point, $GHF$ its diameter. Draw $DF$; construct angle $BOK$=to angle $ACQ$. Now arc $GD$=arc $AQ$, arc $BQ$=arc $DF$=arc$FE$.




$\therefore$ arc $GD$=arc $BF$=arc $AQ$. Arc $BF$ measures angle $BOF$, arc $AQ$ measures angle $ACQ$. But arc $BF=r/R$ arc$BEK$.

$\therefore$ Angle $BOF=r/R$ angle $BOK=r/R$ angle $ACQ$. If $R=3r$, angle $BOF=\frac{1}{3}$angle $ACQ$.

Therefore, by means of a suitable hypocycloid any angle may be divided in any given ratio. This property is applicable to the epicycloid also.

Also solved by the *PROPOSER.*